Math 3235 Probability Theory

6/13/23

Central Limit Theorem.

X_i are i.i.d. r.v.

$\mathbb{E}(X_i) = \mu \qquad Var(X_i) = \sigma^2$

$$Z_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{X_i - \mu}{\sigma}$$

Then

$$Z_N \underset{N\to\infty}{\Longrightarrow} Z$$

where Z is $N(0,1)$.

$$\left(Z_N \Rightarrow N(0,1) \right)$$

$$\mathbb{P}(Z_N \le z) \underset{N\to\infty}{\longrightarrow} \mathbb{P}(Z \le z) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx = \phi(z)$$

$$Z_N = \frac{1}{\sqrt{N}} \sum_i \frac{X_i - \mu}{\sigma}$$

$$\mathbb{E}\left(\frac{X_i - \mu}{\sigma} \right) = 0$$

$$Var\left(\frac{X_i - \mu}{\sigma} \right) = 1$$

$$S_N = \frac{1}{N} \sum_{i=1}^{N} \left(X_i - \mu \right) \xrightarrow{P} 0$$

$$\mathbb{P}(S_N < x) \to \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$T_N = \sum_{i=1}^{N} (X_i - \mu)$$

Typically $T_N << N$!

$$Var\,(T_N) = N\sigma^2$$

$$\sigma_{T_N} = \sqrt{N}\,\sigma$$

It Looks reasonable To consider

$$Z_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{(X_i - \mu)}{\sigma}$$

$$Z_N = \frac{\sqrt{N}}{\sigma}\left(\bar{X} - \mu\right) \simeq N(0,1)$$

$$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

$$\bar{X} \simeq N\left(\mu, \frac{\sigma^2}{N}\right)$$

$$E(\bar{X}) = \mu \qquad Var(\bar{X}) = \frac{\sigma^2}{N}$$

X_i measurement

$E(X_i)$ = True value

$Var(X_i) = \sigma^2$ is Known.

X_i are many measurement and compute $\overline{X}$

$$\mathbb{P}\left(|\overline{X} - \mu| > \delta \right) \leq 0.05$$

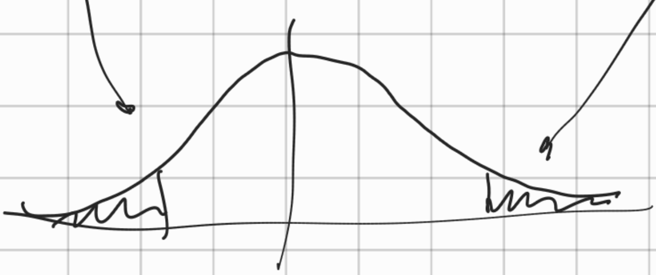
if N is large ($N > 40$)

$$\overline{X} - \mu \approx \mathcal{N}\left(0, \frac{\sigma}{N}\right)$$

$$Z_N = \frac{\overline{X} - \mu}{\sigma/\sqrt{N}} \approx \mathcal{N}(0,1)$$

$$\mathbb{P}\left(|\overline{X} - \mu| > \delta \right) = \mathbb{P}\left(|Z_N| > \frac{\delta\sqrt{N}}{\sigma} \right) =$$

$$\mathbb{P}\left(Z_N \leq -\frac{\delta\sqrt{N}}{\sigma} \right) + \mathbb{P}\left(Z_N > \frac{\delta\sqrt{N}}{\sigma} \right) =$$



$$\mathbb{P}\left(Z \leq -\frac{\delta\sqrt{N}}{\sigma}\right) = 0.025$$

$$\mathbb{P}\left(Z \leq -z_\alpha\right) = \alpha$$

α - critical value

(for Standard Normal)

$$\frac{\delta\sqrt{N}}{\sigma} = z_{0.025} = 1.96$$

$$\delta = 1.96\frac{\sigma}{\sqrt{N}}$$

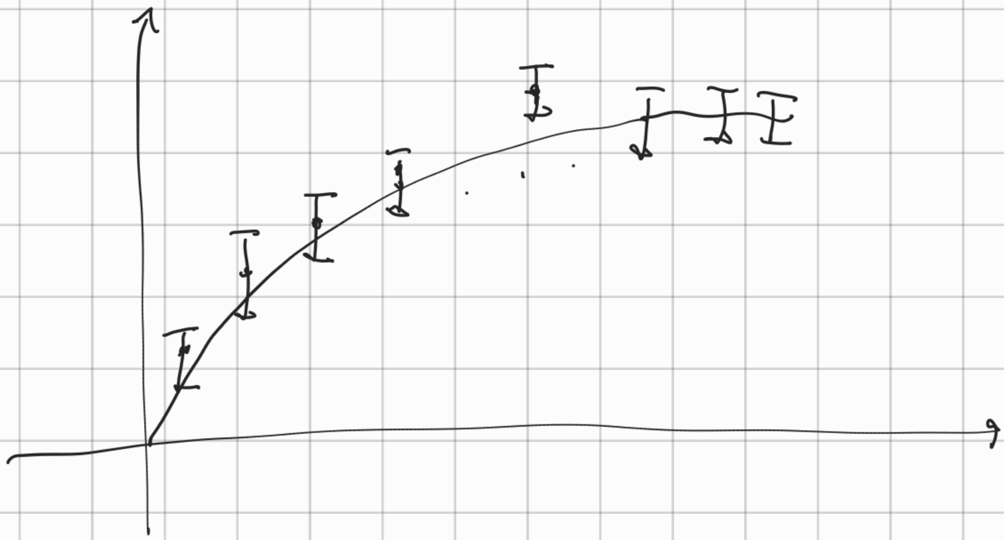
$$\overline{x} - 1.96\frac{\sigma}{\sqrt{N}} \leq \mu \leq \overline{x} + 1.96\frac{\sigma}{\sqrt{N}} \qquad N$$

$$x - 1.96\,\sigma \leq \mu \leq x + 1.96\,\sigma \qquad N = 1$$

if X_i are Normal!

Confidence InTervall

Error bar.



Continuity Theorem

$Z_1 \quad \dots \quad Z_N \quad \dots$

$M_N(t)$ exists for every t

and

$$M_N(t) \to e^{\frac{1}{2}t^2} \qquad \forall t$$

Then

$$Z_N \Rightarrow Z$$

with $Z \sim N(0,1)$.

Proof of C.L.T:

$$Y_i = X_i - \mu \qquad \mathbb{E}(Y_i) = 0$$

$$M_{Y_i}(t) = \mathbb{E}\left(e^{tY_i - t\mu}\right) =$$

$$= e^{-t\mu} M_{X_i}(t)$$

$$Z_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{X_i - \mu}{\sigma}$$

$$M(Z_N) = \mathbb{E}\left(e^{\frac{t}{\sigma\sqrt{N}} \sum_i (X_i - \mu)}\right) =$$

$$= \mathbb{E}\left(e^{\frac{T}{\sigma\sqrt{N}}(X_1 - \mu)}\right)^N$$

$$= \left(M_{Y_1}\left(\frac{t}{\sigma\sqrt{N}}\right)\right)^N =$$

$$M_{Y_1}(0) = 0$$

$$\frac{d}{dt} M_{Y_1}(0) = 0$$

$$\frac{d^2}{dt^2} M_{Y_1}(0) = \mathbb{E}(Y_i^2) = Var(X_i) = \sigma^2$$

$$M_{Y_1}(t) = \left(1 + 2\sigma^2 t^2 + o(t^2)\right)$$

$$M_{Z_N}(t) = \left(M_{Y_1}\left(\frac{t}{\sigma\sqrt{N}}\right)\right)^N =$$

$$\left(1 + \frac{2t^2}{N} + o\left(\frac{t^2}{N}\right)\right)^N$$

$\forall t$

$$\lim_{t \to \infty} M_{Z_N}(t) = e^{2t^2}$$

Characteristic function

$$\phi_X(t) = \mathbb{E}(e^{itX})$$

$$"\phi_X(t) = M_X(it)"$$

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx$$

Fourier Transform.

$$|\phi_X(t)| \leq \int_{-\infty}^{\infty} \left| e^{itx} f_X(x)\, dx \right| \leq$$

$$\leq \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_X(t)\, dt$$